PERGAMON

International Journal of Solids and Structures 40 (2003) 6703–6722

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Effect of strength-modulus correlation on reliability of randomly heterogeneous beams

Sefi Givli, Eli Altus *

Faculty of Mechanical Engineering, Technion, Israel Institute of Technology, Neve-Shaanan, Haifa 32000, Israel

Received 28 January 2003; received in revised form 24 July 2003

Abstract

The strength reliability of linearly elastic, brittle, stochastically heterogeneous beams, is studied on the basis of the weakest link approach. The analysis is formulated by a functional perturbation method, resulting in an analytical solution of the failure probability of the beam. Heterogeneity (material morphology) is random and confined to longitudinal direction only, under Bernoulli assumptions. The problem is statically indeterminate and external loads are not random. The stress field is random and functionally dependent on morphology. In particular, local strength is also a function of modulus. Therefore, the strength reliability of the beam is morphology dependent, both through static indeterminacy and local strength-modulus correlation. The above is also coupled with the probabilistic nature of strength, associated with surface defects and irregularities. The case of single indeterminacy (clamped—simply supported beam) is investigated, for simplicity. It is shown that the strength of the beam is significantly affected by material morphology and that the effect can be either positive (increased strength) or negative, depending on the strength-moduli correlation. For example, for an effective grain size of $L/10$, and a compliance statistical variance of 1/12, the morphology effect on the allowable design load, was found to be in the order of 10%. Calculation of size effect, corresponding to strength, showed a complex, non-classical grain size dependency.

Keywords: Strength; Failure; Reliability; Beam; Elasticity; Probability; Heterogeneity; Morphology; Correlation; Static indeterminacy

1. Introduction

The analysis of structures involving spatially random material properties and/or random geometry has been a major field of research for the past few decades. Interest has emerged from the motivation to find the effective (bulk) elastic modulus in terms of microstructure morphology: Hill (1952) showed that the results of Voigt (1887, uniform strain assumption) and Reuss (1929, uniform stress assumption) are upper and lower bounds for the effective modulus. Finer bounds have been found using variational principles (Hashin and Shtrikman, 1962), perturbation expansion (Kröner, 1986), homogenization and others methods (Mason and Adams, 1999).

* Corresponding author. Tel.: +972-4-8293157; fax: +972-4-8295711/8324533.
E-mail addresses: mesefi@tx.technion.ac.il (S. Givli), altus@tx.technion.ac.il (E. Altus).

0020-7683/$ - see front matter
doi:10.1016/S0020-7683(03)00436-0

In all of the above studies the size of the micro-scale was assumed to be significantly (few orders) smaller than the macro-scale, and representative volume element (RVE) or effective (non-random) bulk properties could be assumed. In recent years, technology progress poses new challenges in predicting the mechanical behavior of stochastic heterogeneous structures for which such assumptions cannot be applied. For example, micro-scale structures used in MEMS/NEMS devices are routinely manufactured from brittle polycrystals (like polysilicon) in which the stress concentration volume is comprised of very few grains. Nanowires and nanotubes which are expected to have important applications in computers, bioengineering and communication, are locally anisotropic, having a non-negligible substructure size. Other examples are porous ceramics (Nanjangus et al., 1995) and bones (Keyak et al., 1994), for which local mechanical properties are random, and strongly dependent upon local density. In many cases, materials are linear elastic, brittle, with dominant probabilistic strength distribution (Jones et al., 1999; Greek et al., 1999; Sharpe et al., 1999; Namazu et al., 2000; Davidge, 1979). Recent studies show that the mechanical response of these structures deviate significantly from the prediction based on effective properties (Mirfendereski et al., 1992; Altus, 2001; Altus and Givli, 2003; Frantziskonis and Breysse, 2003).

Consider as a simple reference problem the strength reliability of a homogeneous Euler–Bernoulli beam, subjected to a non-random external loading. For statically determinate cases, the stresses are independent of moduli. Failure is governed by surface defects and their local stress variations, which are random. Therefore, if the failure probability of a unit length of the beam is obtained experimentally, the strength reliability of the whole beam can be found by direct integration, using the weakest link concept. This type of problem has been studied extensively both for static (Elishakoff, 1983) and dynamic (Lin and Cai, 1995) conditions.

Heterogeneous structures pose new challenges with regard to strength analysis, which can be divided into two categories: (1) Consideration of *local* stress concentrations caused by compatibility requirements. These are common in granular or polycrystal materials where high stress concentrations are found near grain boundaries and other regions of abrupt moduli change. The problem has been addressed mainly by numerical studies (Frantziskonis et al., 1997; Harder, 1999; Starzewski and Stahl, 2000; Barbe et al., 2000a,b). (2) Considering cases of statically indeterminate heterogeneous structures, where reaction forces are randomly coupled with moduli morphology (Altus, 2001) through *external* (global) compatibility conditions.

Both types exist in practice. Their source is similar, i.e., compatibility requirements, but of a different type: the first is local (and is much more difficult to approach analytically), and the second is global. This paper focuses on the heterogeneity effect of the second type only. Moreover, moduli heterogeneity is confined to the beam's longitudinal direction, for simplicity of derivation.

It should be noted that the general problem of strength of heterogeneous media, including morphology effects and various failure criteria, has been studied extensively (for example Herrmann and Roux, 1990; Jeulin, 1993). However, the above compatibility effects have not been considered analytically.

Recently, Altus and Givli (2003) studied the effect of moduli heterogeneity on strength reliability of isotropic beams, where local strength and moduli were independent (non-correlated) material properties. In practice, many materials exhibit strong correlation between these two parameters. For example, in the case of polycrystals, both strength and local modulus depend on crystal orientation. Other examples are porous ceramics or rocks, for which the average tensile strength and Young's modulus are strongly correlated with local density, such that moduli and strength are related through a power law (Kim et al., 2002; Coquard et al., 1994; Nanjangus et al., 1995; Kingery et al., 1976). Interestingly, the power law relation was found even for bones (Keyak et al., 1994; Keller et al., 1989, 1990). In another work (Snead et al., 1995), a strong correlation between strength and moduli was found in ceramics by ion radiation treatment.

This paper studies the effect of moduli morphology on strength reliability of statically indeterminate isotropic structures, and includes the effect of local strength-moduli correlation (SMC). The study is presented in the following order: Section 2 introduces the necessary background material: mathematical notations of convolutions, basic functional operations, relations of stochastic strength, weakest link, functional relation between stress and morphology and the local SMC. Section 3 includes the main theoretical development and analysis by the functional perturbation method (FPM). Analytical examination of limit cases is outlined in Section 4 and full numerical examples are detailed in Section 5, including a consideration of morphology dependent size effect for strength. Major conclusions are summarized in Section 6.

2. Basic relations and operations

The aim of this section is to outline the basic relations that will be needed for the FPM analysis. Section 2.1 reviews some functional operations, and Section 2.2 summarizes the essential beam bending relations from previous study (Altus and Givli, 2003). Section 2.3 discusses the relation between local stiffness and local strength in heterogeneous materials, and Section 2.4 introduces the basic concepts in evaluating the strength of heterogeneous structures that include local SMC, by the weakest link approach.

2.1. Background on functional operations

The Dirac two point singular operator, is defined by (Beran, 1967):

$$\frac{\delta S_1}{\delta S_2} = \frac{\delta S(x_1)}{\delta S(x_2)} = \delta(x_1 - x_2), \tag{2.1a}$$

where $S_1 \equiv S(x_1)$, etc. for convenience. The notation of the δ symbol is common for both the Dirac function and functional derivatives. This duplicity will not cause any confusion. Thus,

$$S(x_1) = \int \delta(x_1 - x_2) S(x_2)\, \mathrm{d}x_2, \qquad S = \delta * S, \tag{2.1b}$$

where (∗) is the convolution symbol. It can be shown that (2.1a) and (2.1b) contain essentially the same definition of δ. Simple integration is written with the aid of a unit function $1(x)$, i.e.,

$$\int S(x)\, \mathrm{d}x = S * 1. \tag{2.2}$$

Additional operations with generalized functions can be found elsewhere (Kanwal, 1983).

Now consider a functional $F\{S(x)\}$ and notice the different notations used here (and throughout) to distinguish between a function ( ) and a functional { } relation. For any function $u(S)$, or functional $F\{S\}$, we have two kinds of differentiations, noted by

$$\frac{\delta F\{S(x)\}}{\delta S(x_1)} \equiv \frac{\delta F}{\delta S_1} \equiv \delta F_{,S_1}, \qquad \frac{\partial u(S)}{\partial S} = \partial u_{,S}, \tag{2.3}$$

where the short notations are used for convenience. Using (2.1) we have also

$$\frac{\delta u(S)}{\delta S_1} = \partial u_{,S} \delta S_{,S_1} = \partial u_{,S} \delta_1, \tag{2.4}$$

where δ and ∂ are used to distinguish between the two types of differentiations, when necessary. When the type of differentiation is clear from the text, no special symbol will be given. F can be functionally expanded by a Taylor series as

$$F\{S_0 + S'\} = F\{S_0\} + F_{,S_1} * S'_1 + \tfrac{1}{2} F_{,S_1 S_2} * *(S'_1 S'_2) + \cdots, \tag{2.5}$$

$F_{,S_1}$ and $F_{,S_1 S_2}$ are first and second (outer) functional derivatives with respect to S_1 and S_2 at S_0. The above expansion is a fundamental tool in the "FPM", used herein. For example, note the following two functionals:

$$F\{S(x)\} = v(x) * S \quad \rightarrow \quad F_{,S_1} = v * \frac{\delta S}{\delta S_1} = v * \delta_1 = v_1, \tag{2.6}$$

$$F\{S(x)\} = (v * S)^{-1} \quad \rightarrow \quad F_{,S_1} = -(v * S)^{-2} u_1 \quad \rightarrow \quad F_{,S_1 S_2} = 2(v * S)^{-3} v_1 v_2, \tag{2.7}$$

where $v_1 = v(x_1)$, $\delta_1 = \delta(x - x_1)$, etc. Mixed derivatives of more complex functional expressions will be also needed in the text. For example, if

$$\phi = F(S_1) * G[\{S_2\}, f_1], \quad f_1 \equiv f(x_1), \tag{2.8}$$

where the integration is over the variable with the common index (here x_1), then

$$\phi_{,S_3} = \frac{\delta F(S_1)}{\delta S_3} * G + F(S_1) * \frac{\delta G}{\delta S_3} = \frac{\partial F(S_3)}{\partial S_3} G[\{S_2\}, f_3] + F(S_1) * \frac{\delta G[\{S_2\}, f_1]}{\delta S_3}, \tag{2.9}$$

where the Dirac property (2.1b) has been used.

2.2. Background on strength and reaction forces

Consider a 1D rod under a distributed stress field $\sigma(x)$. Define the failure probability of a reference element of length ℓ, subjected to a uniform stress $\bar{\sigma}$ by

$$F_\ell(\bar{\sigma}, \ell) = \int_0^{\bar{\sigma}} f_\ell(\sigma', \ell)\, d\sigma', \tag{2.10}$$

where F and f are the failure probability and density, respectively. By the weakest link approach for strength (Davidge, 1979), and for loads of low failure probability (small stresses, high reliability), a power law approximation for $F_\ell(\sigma)$ is possible (Altus and Givli, 2003):

$$F_\ell(\bar{\sigma}, \ell) \cong \left(\frac{\bar{\sigma}}{\sigma_\ell}\right)^\beta, \tag{2.11}$$

where β is identical to the shape parameter when Weibull distribution for F is used. Furthermore, the failure probability of a rod of length L, having a distributed stress field $\sigma(x)$ is

$$F(\sigma(x), L) = \frac{L}{\ell} \int_0^1 F_\ell(\sigma(x), \ell)\, dx. \tag{2.12}$$

Consider now a clamped—simply supported beam (indeterminacy of degree 1), shown schematically in Fig. 1. The internal bending moment is

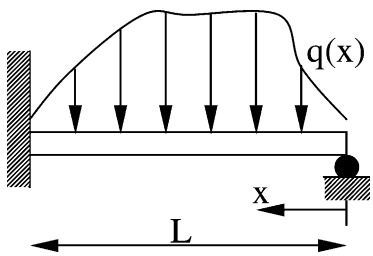

Fig. 1. A beam with one degree of indeterminacy.

$$M(x) = M_R + M_q, \quad M_R = Rx, \;\; M_q = \int_{x_1=0}^{x} q_1(x - x_1)\,\mathrm{d}x_1. \tag{2.13}$$

R is the reaction force at $x = 0$. $M_R(x)$ and $M_q(x)$ are the internal bending moment distributions caused independently by R and $q(x_1)$, respectively. R is found by the compatibility condition:

$$(Mx) * S = 0, \qquad S = [EI]^{-1}, \tag{2.14}$$

where (2.13) has been used. $S(x)$ is the cross-sectional bending compliance, E is Young's modulus and I is the appropriate cross-sectional inertia term. For convenience, normalized quantities will be used throughout, i.e.,

$$S \to \frac{S}{\langle S \rangle}, \qquad x \to \frac{x}{L}, \qquad \langle S \rangle \to 1. \tag{2.15}$$

Nevertheless, whenever the average compliance is used, we will keep the notation $S = \langle S \rangle$ instead of $S = 1$ for clarity. The beam morphology is considered as statistically homogeneous, therefore $\langle S \rangle$ stands for both the spatial and ensemble averages, and is not a function of x. From (2.13) and (2.14), a functional expression for the reaction force is obtained:

$$R = -\frac{[M_q x] * S}{x^2 * S}, \tag{2.16}$$

which shows that R is independent of moduli for a homogeneous beam.

2.3. Local stiffness–strength correlation

When material exhibits strong correlation between Young's modulus and strength, (2.11) is generalized to

$$F_\ell \cong \left(\frac{\bar{\sigma}}{\sigma_S(S)} \right)^\beta, \tag{2.17}$$

where $\sigma_S(S)$ can be interpreted as an estimate for the average failure stress of a reference element with cross-sectional compliance S. For example, if a linear relation between strength and Young's modulus is assumed, as discussed above for porous ceramics, we write

$$\sigma_S(S) \propto E = \frac{1}{S}, \tag{2.18}$$

and (2.17) becomes

$$F_\ell = \left(\frac{\bar{\sigma}}{\sigma_0 \cdot \frac{1}{S}} \right)^\beta = S^\beta \left(\frac{\bar{\sigma}}{\sigma_0} \right)^\beta, \quad \sigma_0 = \sigma_S(S = 1). \tag{2.19}$$

For a general relation between local strength and modulus, it is convenient to write

$$F_\ell = \left(\frac{\bar{\sigma}}{\sigma_0 \cdot \bar{\sigma}_0(S)} \right)^\beta = \theta(S) \cdot \left(\frac{\bar{\sigma}}{\sigma_0} \right)^\beta, \quad \theta = [\bar{\sigma}_0(S)]^{-\beta}, \tag{2.20}$$

where σ_0 is independent of S, and θ, $\bar{\sigma}_0$ are non-dimensional functions of S such that

$$\bar{\sigma}_0(S = \langle S \rangle) = \theta(S = \langle S \rangle) = 1. \tag{2.21}$$

From (2.20) and (2.21) we see that σ_0 can be conceived as the average failure stress of a reference element which possesses a uniform modulus $\langle S\rangle$. $\bar{\sigma}_0$ is a relative measure of strength

$$\bar{\sigma}_0(S)=\frac{\sigma_0(S)}{\sigma_0(\langle S\rangle)}. \tag{2.22}$$

$\theta(S)$ has no direct physical interpretation, but holds some algebraic advantages that will lead to more concise expressions. Thus, θ will be used in the following, leaving $\bar{\sigma}_0$ for the final expressions.

2.4. Weakest link approach involving local strength-modulus correlation (SMC)

The expressions in (2.17) and (2.20) indicate that if the local stiffness of the basic element is known, its failure probability can be found explicitly. Using (2.20) and the weakest link principle, the failure probability of the whole beam for a given (i.e. arbitrary, not random) stiffness function $S(x)$, yields

$$F_{\mathrm{b}}|_{S(x)}=\frac{L}{\ell}\left(\theta(S)\cdot\left(\frac{\sigma}{\sigma_0}\right)^{\beta}\right)*1=\frac{L}{\ell}\cdot\theta(S)*\left(\frac{\sigma}{\sigma_0}\right)^{\beta}. \tag{2.23}$$

In the general case, when the stiffness is unknown explicitly (i.e. random), the failure probability of the whole beam is just the average of all probabilities of all realizations of S. Therefore,

$$F_{\mathrm{b}}=\langle F_{\mathrm{b}}|_S\rangle=\frac{L}{\ell}\left\langle\theta(S)*\left(\frac{\sigma}{\sigma_0}\right)^{\beta}\right\rangle. \tag{2.24}$$

Notice that in the particular case of static determinacy, since stresses are calculated from equilibrium considerations only (independent of stiffness), we can write

$$F_{\mathrm{b}}^{(\mathrm{det})}=\frac{L}{\ell}\left[\langle\theta(S)\rangle\cdot\left(\frac{\sigma}{\sigma_0}\right)^{\beta}\right]*1. \tag{2.25}$$

Also, since $\langle\theta(S)\rangle$ is considered as a constant with respect to the integration, (2.25) can be written in the familiar form

$$F_{\mathrm{b}}^{(\mathrm{det})}=\frac{L}{\ell}\left(\frac{\sigma}{\sigma_0^*}\right)^{\beta}*1,\quad \sigma_0^*=\sigma_0\cdot\langle\bar{\sigma}_0^{-\beta}\rangle^{-1/\beta}=\sigma_0\cdot\langle\theta\rangle^{-1/\beta} \tag{2.26}$$

where σ_0^* can be conceived as the average tensile strength of all reference elements with length ℓ that posses a uniform (yet random) stiffness.

3. Strength of statically indeterminate heterogeneous beam

3.1. General solution

Consider the case of statically indeterminate heterogeneous beam shown in Fig. 2, for which the local strength reliability and the compliance are related by (2.20). The reaction force R is a functional of the compliance morphology S by (2.16). From (2.24) the general functional form of the beam failure probability is

$$F_{\mathrm{b}}=F_{\mathrm{b}}[\{\theta(S)\},R\{S\}]. \tag{3.1}$$

F_b is a functional of θ and a function of R, whereas θ is a function of $S(x)$ and R is a functional of $S(x)$. In case where local strength and modulus are independent, θ is not a function of S and (3.1) reduces to

$$F_b = F_b(R\{S\}), \tag{3.2}$$

and it is possible to write

$$F_b = F_b|_{\langle R\rangle} + \tfrac{1}{2}\partial F_{b,RR}|_{\langle R\rangle} \cdot \langle R'^2\rangle + \cdots. \tag{3.3}$$

Then, $\langle R\rangle$ and $\langle R'^2\rangle$ can be calculated separately from (2.16).

It is clear that this separation cannot be applied here and the FPM has to be generalized. For any given realization, we have

$$F_b|_S = \frac{L}{\ell}\left(\theta(S) * \left(\frac{\sigma}{\sigma_0}\right)^{\beta}\right) = C \cdot \theta(S) * \psi(M\{S\}), \tag{3.4a}$$

where

$$\psi = |M|^{\beta}, \qquad C \equiv \frac{L}{\ell}\cdot\left(\frac{B}{\sigma_0}\right)^{\beta}. \tag{3.4b}$$

B is the geometry factor, which relates the local surface stress to bending moment by the elementary Euler beam relations. ψ is introduced here for mathematical convenience.

Note that Eqs. (3.4) consider failure due to near surface stresses only, appropriate for brittle materials subjected to bending. In addition, the analysis is limited to a symmetric cross-section geometry, in which the maximum and minimum bending stresses are found at equal distances, but on opposite sides from the center of gravity.

In the general case of beams of random compliance, the failure probability of the beam is the average failure probability of the ensemble, i.e.,

$$F_b = \langle F_b|_S\rangle = C \cdot \langle \theta(S) * \psi(M\{S\})\rangle \equiv F_b\{S\}. \tag{3.5}$$

Thus, (3.5) exhibits a functional averaging with respect to $S(x)$, and not a parametric averaging. Using the relation

$$S(x) = 1 + S'(x) \;\Rightarrow\; \frac{\delta}{\delta S} = \frac{\delta}{\delta S'}, \tag{3.6}$$

expansion of (3.5) into a series near $\langle S\rangle$ (i.e. $\langle S'\rangle = 0$) is written as

$$F_b\{S'\} = F_b|_{S'=0} + \delta F_{b,S'}|_{S'=0} * \langle S'\rangle + \tfrac{1}{2}\delta^2 F_{b,S'S'}|_{S'=0} ** \langle S'S'\rangle + \cdots \tag{3.7}$$

in which the second term vanishes identically. Terms in the order of $\langle S'S'S'\rangle$ and higher are neglected. The first term in (3.7) is the failure probability of the non-random (homogeneous) case, and the third is a contribution of morphology-strength coupling, originated from the indeterminacy and SMC.

From (3.5) we obtain

$$\delta F_{b,S'_1} = C \cdot \left(\delta\theta_{,S'_1} * \psi + \theta * \delta\psi_{,S'_1}\right) = C \cdot \left(\partial\theta_{1,S'_1} \cdot \psi_1 + \theta * \delta\psi_{,S'_1}\right), \tag{3.8}$$

where the index notations are consistent with the definitions in Section 2. A second functional derivative yields three types of expressions

$$\delta^2 F_{b,S'_1S'_2} = C \cdot \left[\partial^2\theta_{1,S'_1S'_1} \cdot \psi_1 \cdot \delta(x_1 - x_2) + \partial\theta_{,S'_1} \cdot \delta\psi_{1,S'_2} + \partial\theta_{,S'_2} \cdot \delta\psi_{2,S'_1} + \theta * \delta^2\psi_{,S'_1S'_2}\right]. \tag{3.9}$$

From (3.4b) we have that

$$\delta\psi_{,S_1'} = \partial\psi_{,|M|} \cdot \partial|M|_{,M} \cdot \delta M_{,S_1'}. \tag{3.10}$$

To calculate (3.10), note first that

$$|M| = M \cdot \mathrm{sign}(M) \tag{3.11}$$

and

$$\mathrm{sign}(M) = 2H(M) - 1, \tag{3.12}$$

where H is the Heaviside (unit step) function at $M = 0$. Thus, from (3.11) and (3.12) we obtain

$$\partial|M|_{,M} = \mathrm{sign}(M) + 2\delta(M) \tag{3.13}$$

where δ is the Dirac function. However, since $\delta(M)$ is non-zero only when $M = 0$, its contribution to the following convolutions vanishes. Therefore, we disregard the second term of (3.13) and write

$$\delta\psi_{,S_1'} = \mathrm{sign}(M) \cdot \beta \cdot |M|^{\beta-1} \cdot \delta M_{,S_1'}. \tag{3.14}$$

Evaluating the variation of (3.14) with respect to $S'(x_2)$ yields

$$\delta^2\psi_{,S_1'S_2'} = \beta(\beta-1) \cdot |M|^{\beta-2} \cdot \delta M_{,S_1'} \cdot \delta M_{,S_2'} + \mathrm{sign}(M) \cdot \beta \cdot |M|^{\beta-1} \cdot \delta^2 M_{,S_1'S_2'}. \tag{3.15}$$

Now, in order to calculate (3.7) using (3.15) and (3.14), we have to find first the expressions $\delta M_{,S'}$ and $\delta^2 M_{,S'S'}$ at $S' = 0$. From (2.13) and (2.16)

$$M\{S'\} = M_q - \frac{(M_q \cdot x) * (1+S')}{x^2 * (1+S')} \cdot x \tag{3.16}$$

which yields

$$\delta M_{,S_1'} = \left[-\frac{M_{q1} \cdot x_1}{x^2 * (1+S')} + \frac{(M_q \cdot x) * (1+S') \cdot x_1^2}{(x^2 * (1+S'))^2} \right] \cdot x \tag{3.17}$$

and

$$\delta^2 M_{,S_1'S_2'} = \left[\frac{M_{q1} \cdot x_1 \cdot x_2^2 + M_{q2} \cdot x_2 \cdot x_1^2}{[x^2 * (1+S')]^2} - 2\frac{(M_q \cdot x) * (1+S') \cdot x_1^2 \cdot x_2^2}{[x^2 * (1+S')]^3} \right] \cdot x. \tag{3.18}$$

For simplicity, a single notation x is used for the location parameter in the above convolutions terms, while formally different signs (x_4, x_5, etc.) have to be assigned. Evaluating (3.16)–(3.18) at $S' = 0$ yields

$$M_h = M_q - \frac{(M_q \cdot x) * 1}{x^2 * 1} \cdot x \tag{3.19}$$

and

$$\delta M_{,S_1'}|_{S'=0} = -\frac{M_{h_1} \cdot x_1}{x^2 * 1} \cdot x, \tag{3.20}$$

$$\delta^2 M_{,S_1'S_2'}|_{S'=0} = \left[\frac{M_{h_1} \cdot x_1 \cdot x_2^2 + M_{h_2} \cdot x_2 \cdot x_1^2}{[x^2 * 1]^2} \right] \cdot x, \tag{3.21}$$

where (3.19) has been used in (3.20) and (3.21). Finally, inserting (3.9), (3.14), (3.15), (3.20) and (3.21) into (3.7) we obtain

$$F_b = C\left(F_b^{(0)} + F_b^{(1)} + F_b^{(2)} + F_b^{(3)}\right) = F_b^{hom} + F_b^{het}, \tag{3.22}$$

where

$$F_b^{(0)} = \theta|M_h|^\beta * 1, \tag{3.23a}$$

$$F_b^{(1)} = \tfrac{1}{2}\partial\theta_{,ss} \cdot \langle S'^2\rangle \cdot |M_h|^\beta * 1 \tag{3.23b}$$

$$F_b^{(2)} = \tfrac{9}{2}\theta \cdot \Big[\beta(\beta-1)\cdot(|M_h|^{\beta-2} * x^2)\cdot(M_{h_1}x_1)\cdot(M_{h_2}x_2) + 2\beta(\mathrm{sign}(M_h)|M_h|^{\beta-1} * x)\cdot(M_{h_1}\cdot x_1\cdot x_2^2)\Big] ** \langle S_1'S_2'\rangle, \tag{3.23c}$$

$$F_b^{(3)} = -3\beta\cdot\partial\theta_{,s}\cdot(\mathrm{sign}(M_{h_1})\cdot|M_{h_1}|^{\beta-1}M_{h_2}x_1x_2) ** \langle S_1'S_2'\rangle, \tag{3.23d}$$

$$F_b^{hom} = C\left(F_b^{(0)} + F_b^{(1)}\right), \qquad F_b^{het} = C\left(F_b^{(2)} + F_b^{(3)}\right). \tag{3.23e}$$

θ and its derivatives are evaluated at $\langle S\rangle$. The relation

$$x^2 * 1 = \tfrac{1}{3}, \tag{3.24}$$

and the symmetry of $\langle S_1'S_2'\rangle$ have been used in (3.23c) and (3.23d).

The expressions in (3.22) and (3.23) provide an explicit solution for any given morphology, through the two point correlation function $\langle S'S'\rangle$. The different terms of (3.22) are ordered in increasing complexity with respect to morphology: $F_b^{(0)}$ is independent of S, and corresponds to the failure probability of the homogeneous ensemble case (elastic modulus is not random). $F_b^{(1)}$ is independent of morphology, and as will be shown in the next section, it is related to the homogenized (very large or very small grain size) case. $F_b^{(2)}$ contains the morphology effect on failure probability for a material *without* strength-modulus correlation (SMC). Finally, $F_b^{(3)}$ describes the net SMC contribution.

For later use, it is important to examine the sign of $F_b^{(2)}$ in (3.23c). Its first term is positive by symmetry arguments. The second is positive except that $\mathrm{sign}(M_{h_1})M_h$, may be negative. However, in most cases the major contribution to the convolutions comes from regions where $(x - x_1) \ll 1$, and therefore, $\mathrm{sign}(M_{h_1})M_h \sim \mathrm{sign}(M_{h_1})M_{h_1} > 0$. Consequently, the second term is also positive and $F_b^{(2)}$ itself is positive. It means that non-uniform moduli always increase the probability of failure when no SMC is present. This is *not* true for $F_b^{(3)}$, as will be shown in the following.

3.2. Another perspective on the form of the general solution (3.22)

The solution in (3.22) and (3.23) is an approximation. Its accuracy depends on the number of terms taken in the multiple point probability series for the modulus $S(x)$. The series is of the form

$$F_b = F_{b0} + F_{b1} * \langle S_1'\rangle + F_{b2} ** \langle S_1'S_2'\rangle + F_{b3} *** \langle S_1'S_2'S_3'\rangle + \cdots, \tag{3.25}$$

where F_{bi} are functions of $\langle S\rangle$ and $x_1, x_2, \ldots, x_i$. Usually, measurements of averages like $\langle S_1'S_2'\cdots S_n'\rangle$ is an enormous task for large n, and it is common to collect up to two points data ($n = 2$) from practical purposes. However, when the heterogeneity is not sufficiently small, the first three terms in (3.25) may not be enough. To improve the accuracy without involving higher point probabilities, we notice first that by way of measuring S_1' and $S_1'S_2'$, we have at our disposal the associated density functions $p_1(S)$ and $p_2(S_1, S_2)$, i.e.,

$$\langle S_1'\rangle = \int p_1 S_1' \cdot \mathrm{d}S_1', \qquad \langle S_1'S_2'\rangle = \int p_2 S_1'S_2' \cdot \mathrm{d}S_1'\,\mathrm{d}S_2'. \tag{3.26}$$

Knowing p_1 and p_2, we can find the averages of *any* function of S_1 and S_2. Specifically, we have that

$$\langle S_1'^k \rangle = \int p_1 S_1'^k \cdot \mathrm{d}S_1', \qquad \langle S_1'^k S_2'^m \rangle = \int p_2 S_1'^k S_1'^m \cdot \mathrm{d}S_1' \, \mathrm{d}S_2'. \tag{3.27}$$

Therefore, terms like $\langle S'^n \rangle$ and $\langle S'^k S'^n \rangle$ contain only one and two point probability data, respectively. However, they are included in the nth point ($\langle S_1' S_2' \cdots S_n' \rangle$) and $n+k$ point ($\langle S_1' S_2' \cdots S_{n+k}' \rangle$) probability terms of (3.25).

Now re-examining the initial form of F_b in (3.4). It is composed of terms of the form

$$(S * f)^m \quad \text{and} \quad \theta(S) * f_\theta, \tag{3.28}$$

which can be expanded to a series near $\langle S \rangle$ and averaged, as

$$\langle (S * f)^m \rangle = (\langle S \rangle * f)^m + A_1(f_1 * \langle S_1' \rangle) + A_2(f_1 f_2 * * \langle S_1' S_2' \rangle) + \cdots \tag{3.29}$$

and

$$\langle (\theta * f) \rangle = \langle \theta \rangle * f, \tag{3.30}$$

where

$$\langle \theta \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n \theta}{\partial S^n} \right)_{S=\langle S \rangle} \cdot \langle S'^n \rangle. \tag{3.31}$$

The difference between (3.29) and (3.31) is that the terms in (3.29) contain increasing number of point probabilities, while from (3.27), (3.31) contains only one point statistical data.

Returning now to (3.22) and (3.23) and comparing with (3.31), one can show by further expansion that $F_\mathrm{b}^{(0)}$ and $F_\mathrm{b}^{(1)}$ are just the first two terms in the expansion of $\langle \theta \rangle$. It is thus fruitful to include the rest of the terms in the solution by writing

$$F_\mathrm{b}^{\mathrm{hom}} = C \langle \theta \rangle |M_h|^\beta * 1. \tag{3.32}$$

Using (3.32) instead of (3.23a,b), the overall solution is more accurate, without the need for higher point probability data. In general, the key to the above modification is the presence of "degenerated" terms such as $\langle S_1' S_2'^2 S_4' \rangle$ (where S_3' does not appear) in the series approximation. Mathematically, it is a part of the 4th order term of the expansion, but physically, it belongs to the 3-point probability function.

To demonstrate the difference between (3.32) and (3.23a,b), recall that in order to have the average $\langle \theta(S) \rangle$, we need the whole "one point probability function" $p_1(S)$ and not just the average and variance of S. Therefore, we consider a particular example of a uniform distribution of S in the interval 0.5–1.5, i.e.,

$$p_1(S) = \begin{bmatrix} 1 & (0.5 < S < 1.5) \\ 0 & (\text{otherwise}) \end{bmatrix}. \tag{3.33}$$

for which $\langle S \rangle = 1$ and $\langle S'^2 \rangle = 1/12$. In Fig. 2, the exact $\langle \theta \rangle$ and its two terms approximation are plotted as functions of β. The importance of using the exact form is evident for large values of β. A similar result is expected for any other common pdf.

3.3. Effect of strength-modulus correlation (SMC) on reliability: general considerations

The main goal of the present study is to find what is the effect of SMC (i.e., $F_\mathrm{b}^{(3)}$) on the overall failure probability (reliability) of the beam. Two relations are of interest

$$F_\mathrm{b}^{(3/h)} = \frac{F_\mathrm{b}^{(3)}}{F_\mathrm{b}^{(\mathrm{hom})}} \quad \text{and} \quad F_\mathrm{b}^{(3/2)} = \frac{F_\mathrm{b}^{(3)}}{F_\mathrm{b}^{(2)}}. \tag{3.34}$$

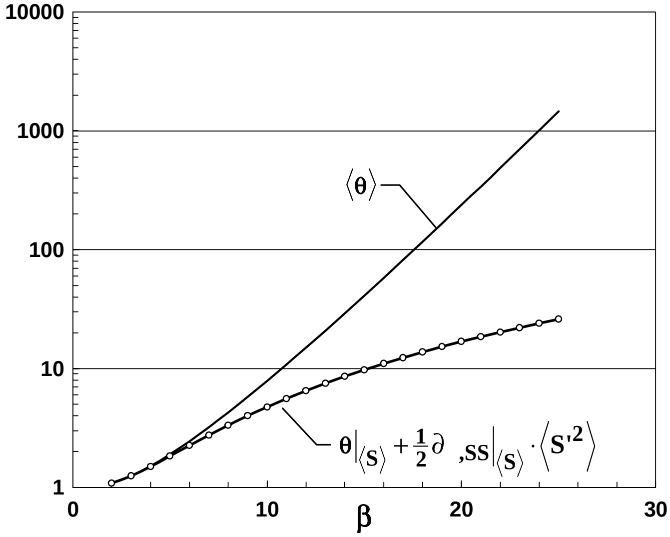

Fig. 2. $\langle\theta\rangle$ calculated from a uniform distribution function (3.33), compared with a two-term approximation. $\langle S'^2\rangle = 1/12$.

The first is the SMC effect relative to the basic homogeneous case and the second is relative to the heterogeneity effect without SMC. Notice that both of them are invariant to the magnitude of the external load (through M_h), and sensitive only to its shape. By the same arguments which led to the conclusion that the sign of $F_b^{(2)}$ is positive, it is easily seen that the sign of $F_b^{(3)}$ in (3.23d) is governed by the sign of $(-\partial\theta/\partial S)$. From (2.20) we have

$$\theta_{,S} = -\beta\bar{\sigma}_0^{-\beta-1}\bar{\sigma}_{0,S} \quad \rightarrow \quad \mathrm{sign}(\theta_{,S}) = -\mathrm{sign}(\bar{\sigma}_{0,S}). \tag{3.35}$$

Commonly, the strength decreases with increasing compliance. Therefore, the $\theta_{,S}$ is positive, leading to an *increased* reliability.

The above can be explained intuitively as follows. For the same load and geometry, increasing the stiffness of the beam locally (i.e., in the vicinity of some point only) is expected to increase the local stresses. This means, that if we examine the beams at a certain point x, there will be a positive correlation between the stiffness and the stress at that point. Now, if the strength is positively correlated with the stiffness (as most materials do), it is negatively correlated with the compliance. This means that the strength is more ''effectively dispersed'', giving more ''strength weight'' to neighborhoods of higher stresses, as compared to the case where it is un-correlatively spread. Therefore, the overall effect of $F_b^{(3)}$ is by *decreasing* the failure probability.

Since $F_b^{(2)}$ and $F_b^{(3)}$ have opposite signs, the overall heterogeneity effect on strength is not clear, and depends on the particular structure and loading. Examples will be given in the next section.

3.4. Size effect of failure probability (reliability)

It was shown (Altus, 2001) that for loading functions f and g, in the case of small moduli correlation distance:

$$f_1 * \langle S_1' S_2' \rangle * g_2 \cong \lambda \langle S'^2 \rangle (f_1 * g_1), \quad \lambda = \frac{\langle S'S' \rangle_{(h)} * 1}{\langle S'^2 \rangle}, \quad h = |x_2 - x_1|. \tag{3.36}$$

The length λ is proportional to the "grain size", where "grain" has a generalized (more abstract) meaning of a characteristic distance in which a statistical moduli correlation exists. For example, if we have a simple case of a beam composed of equal size unidirectional elements, where the moduli inside each element is uniform, and there is no moduli correlation between any two elements, then λ is *exactly* the size of the elements. Therefore, in a random media, λ is loosely equal (of the same order) to the grain size.

Now consider two sets of beams of the same material and cross-section and lengths L_1 and L_2. Apply two load distributions such that $M_h^{(L_1)}$ and $M_h^{(L_2)}$ are of the same shape and have the same magnitude (i.e., their maximum value is also the same). Then, using (3.23) and (3.4), the ratio between the failure probabilities of the two beams is of the form

$$\frac{F_{\mathrm{b}}^{(L_1)}}{F_{\mathrm{b}}^{(L_2)}} = \left(\frac{L_1}{L_2}\right)\frac{1+\phi**\langle S'S'\rangle^{(L_1)}}{1+\phi**\langle S'S'\rangle^{(L_2)}}, \tag{3.37}$$

where $\phi = \phi(\{M_h\}, x_1, x_2)$ is a functional of the loading shape. Without going into more details (example will be given later), it is easily verified that

$$\left.\frac{F_{\mathrm{b}}^{(L_1)}}{F_{\mathrm{b}}^{(L_2)}}\right|_{\lambda=0} = \left.\frac{F_{\mathrm{b}}^{(L_1)}}{F_{\mathrm{b}}^{(L_2)}}\right|_{\lambda\to\infty} = \left(\frac{L_1}{L_2}\right) \tag{3.38}$$

which is identical to the classical "Weibull type" size effect for brittle materials. However, for a finite λ, (3.37) depends on the grain size too. With this respect, we can consider two possibilities. In the first, the *relative* size for the two beams remains the same, therefore $\langle S'S'\rangle^{(L_1)} = \langle S'S'\rangle^{(L_2)}$ and (3.37) reduces to (3.38). In the second, which is the more practical case, the two beams are cut from the same piece of material, and their *absolute* grain size (λL) is the same, i.e.,

$$\lambda^{(L_2)} = \frac{L_1}{L_2}\lambda^{(L_1)}. \tag{3.39}$$

More on the meaning of λ is found in Section 4.3. Inserting (3.39) in (3.37), we obtain for $\lambda \ll 1$ the form

$$\frac{F_{\mathrm{b}}^{(L_1)}}{F_{\mathrm{b}}^{(L_2)}} = \left(\frac{L_1}{L_2}\right)\frac{1+A\lambda^{(L_1)}}{1+A\lambda^{(L_1)}(L_1/L_2)}, \tag{3.40}$$

where A is a functional which depends on morphology and loading distribution. Therefore, the size effect decreases when A is positive, and vice versa.

3.5. Heterogeneity effects on design loads

The heterogeneity effect can be "transferred" from the failure probability space into the allowable loading space, i.e., finding how much the magnitude of the allowable (design) external loads is changed for a given failure probability, due to material heterogeneity. From (3.23) we have

$$F_{\mathrm{b}} = (k_{\mathrm{hom}} + k_{\mathrm{het}})q_0^{\beta}, \tag{3.41}$$

where q_0 is the loading magnitude corresponding to a failure probability F_{b}. k_{hom} and k_{het} are the relative contributions due to strength randomness and moduli heterogeneity, respectively. Define

$$\eta = \left(\frac{k_{\mathrm{het}}}{k_{\mathrm{hom}}}\right) \tag{3.42}$$

as the ratio which reflects the relative contribution of morphology on failure probability. Then, compare two calculations for the same problem, with and without the heterogeneity part for the same reliability prediction, i.e.,

$$k_{\text{hom}}(q_0^{\text{hom}})^{\beta} = (k_{\text{hom}} + k_{\text{het}})q_0^{\beta}. \tag{3.43}$$

Then, from (3.41) and (3.42), the morphology effect on the magnitude of the allowable load is obtained by

$$\frac{q_0}{q_0^{\text{hom}}} = \left(\frac{k_{\text{hom}}}{k_{\text{hom}} + k_{\text{het}}}\right)^{1/\beta} = (1+\eta)^{-1/\beta}. \tag{3.44}$$

4. Insight by limit cases

The failure probability of the beam can be calculated using (3.23) for any given external load (through M_h), provided that the correlation function $\langle S'S' \rangle$ is known explicitly. Before going into explicit examples, some general features of the solution are explored, which correspond to certain morphological limits.

4.1. Homogeneous ensemble

Consider the simplest case of a homogeneous ensemble, i.e. all beams (realizations) in the ensemble are uniform, and posses the *same* stiffness. Then,

$$\langle S_1'S_2' \rangle = \langle S'^2 \rangle = 0 \;\;\Rightarrow\;\; \langle \theta \rangle = \theta(\langle S \rangle) = 1 \;\;\Rightarrow\;\; F_{\text{b}}^{(1)} = F_{\text{b}}^{(2)} = F_{\text{b}}^{(3)} = 0 \tag{4.1}$$

and the failure probability (3.23) reduces to

$$F_{\text{b}} = C \cdot |M_h|^{\beta} * 1 = CF_{\text{b}}|_{S=\langle S \rangle} \tag{4.2}$$

which is exact.

4.2. Uncorrelated local strength and modulus

When local strength and modulus are independent material properties, θ is independent of S. From (2.21) we obtain that

$$\theta = 1. \tag{4.3}$$

Inserting (4.3) into (3.23), $F_{\text{b}}^{(1)}$ and $F_{\text{b}}^{(3)}$ vanish identically, and the failure probability of the beam reduces to

$$F_{\text{b}} = C \cdot \Big(|M_h|^{\beta} * 1 + \tfrac{9}{2}\Big[\beta(\beta-1) \cdot (|M_h|^{\beta-2} * x^2) \cdot (M_{h_1}x_1) \cdot (M_{h_2}x_2) + 2\beta(\text{sign}(M_h)|M_h|^{\beta-1} * x) \cdot (M_{h_1} \cdot x_1 \cdot x_2^2)\Big] * * \langle S_1'S_2' \rangle \Big). \tag{4.4}$$

The expression in (4.4) is identical to the one obtained in a previous study (Altus and Givli, 2003), although by a different approach.

4.3. Small correlation length

Here we examine the case where the morphological correlation length is relatively small, i.e., $\langle S'S' \rangle$ contributes to the integrals only when $|x_1 - x_2|$ is small enough. By (3.36), when $\lambda \ll 1$, the failure probability of the beam given by (3.22) yields

$$F_b^{(0)} + F_b^{(1)} = \langle \theta \rangle \cdot |M_h|^\beta * 1, \tag{4.5a}$$

$$F_b^{(2)} = \frac{1}{2} \cdot \lambda \langle S'^2 \rangle \left[9 \cdot \beta(\beta - 1) \cdot (|M_h|^{\beta-2} * x^2) \cdot (M_h^2 * x^2) + 18 \cdot \beta \cdot (\text{sign}(M_h) \cdot |M_h|^{\beta-1} * x)(M_h * x^3) \right], \tag{4.5b}$$

$$F_b^{(3)} = -3 \cdot \lambda \langle S'^2 \rangle \cdot \beta \cdot \partial\theta_{,s}\big|_{\langle S \rangle} \cdot (|M_h|^\beta * x^2), \tag{4.5c}$$

where the relation (2.21) has been used. The heterogeneity effect is proportional to the grain size and to the compliance variance.

4.4. Infinitely long correlation length

The case of infinitely long correlation length is related to uniform beams. Yet, different from the "homogeneous ensemble" case, the value of the modulus of each beam (realization) is random. In this case, stresses (reaction force) in the beam are immaterial, i.e. equal for all realizations. Therefore, the problem can be considered as statically determinate, and using (2.25) we obtain

$$F_b = F_b^{\text{hom}} = C \langle \theta \rangle \cdot (M_h^\beta * 1). \tag{4.6}$$

Notice that the result in (4.6) is identical to (3.32). This means that the failure probability of the beam takes the same value at both limits $\lambda \to \infty$ and $\lambda \to 0$, and should posses at least one extremum value for some finite grain size.

4.5. Very large β values

Examining the terms in (3.23), (3.31) and (3.32), it is seen that when β is very large (say, greater than 20), the relative weights of the terms of F_b receive a distinct hierarchy. First note that

$$(|M_h|^\beta * 1)_{\beta \gg 1} \propto \left[\frac{(|M|_{\max})^\beta}{\beta} \right], \tag{4.7}$$

where $|M|_{\max}$ is the largest value of the bending moment (i.e., the magnitude). Now examine the case $\theta = S^\beta$ (2.18), for which

$$(F_b^{\text{hom}})_{\beta \gg 1} \propto \left[\frac{(|M|_{\max})^\beta}{\beta} A^\beta \right], \tag{4.8a}$$

where A is a parameter greater than 1. Also,

$$F_b^{(2)} \beta \gg 1 \propto \left[\beta (|M|_{\max})^\beta \right], \tag{4.8b}$$

$$F_b^{(3)} \beta \gg 1 \propto \left[\beta (|M|_{\max})^\beta \right]. \tag{4.8c}$$

Then, it is clear that for large β values, (4.8a) becomes dominant due to A, although none of the terms of (3.23) vanishes.

Contrary to the above, when the strength is not correlated to the compliance, $A = 1$ and (4.8c) vanishes by (3.23d). Then, $F_b^{(2)}$ is the dominant part. Therefore, the SMC plays a major role in the structure reliability design.

5. Numerical examples

The effect of stiffness heterogeneity given by (3.32) and (3.23) reveals a complex and non-linear interaction between material morphology, local strength-stiffness correlation, and loading geometry. For demonstration, three selected load distributions $q(x)$ for the clamped-simply supported (Fig. 1) problem are solved numerically: a concentrated moment at $x = 0$, a uniformly distributed load, and a power form. The distribution shapes of the internal bending moments M_h for the corresponding homogeneous beams are

$$M_h^{(1)} = 3x - 2, \qquad M_h^{(2)} = 4x^2 - 3x, \quad M_h^{(3)} = \tfrac{4}{3}x^{10} - \tfrac{1}{3}x, \tag{5.1}$$

respectively. In all cases, a linear relation between strength and stiffness, as given in (2.18), is used.

5.1. Small correlation length

Inserting (5.1) into (4.6) and integrating, we obtain the effect of compliance heterogeneity on failure probability of the beam as a function of β, λ, $\langle S'^2 \rangle$ and $\langle \theta \rangle$. Recall that in order to obtain $\langle \theta(S) \rangle$, we need more than the average and variance of S. Therefore, we consider again the probability density $p_1(S)$ as in (3.33) and $\lambda = 1/10$.

Using (4.5) and (3.44), the ratios F_b/F_b^{hom} and q_0/q_0^{hom} are plotted in Figs. 3 and 4 as a function of β for the three loading cases (5.1). The morphology effect is the smallest for the concentrated moment $M_h^{(1)}$, and largest for the high power distribution $M_h^{(3)}$. To understand why, recall that the contribution to the beam failure from any segment of the beam is proportional to σ^β (2.17). Therefore, a small region along the beam,

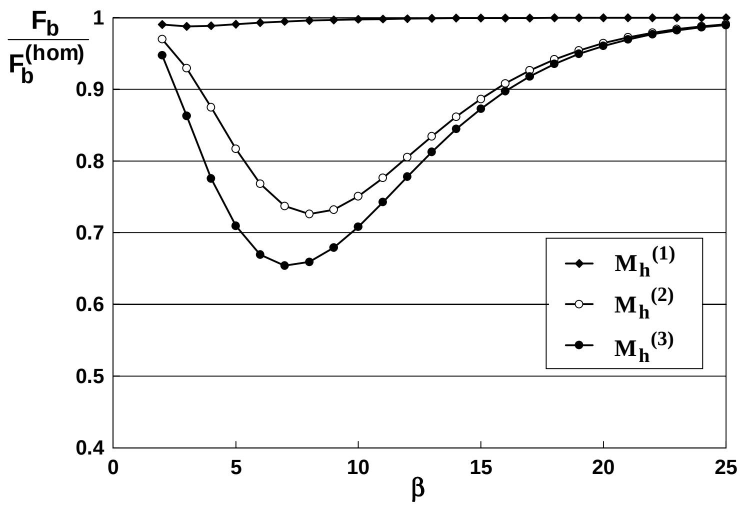

Fig. 3. Effect of stiffness heterogeneity on the beam failure probability for the three loads in (5.1). $\langle S'^2 \rangle = 1/12$, $\lambda = 0.1$.

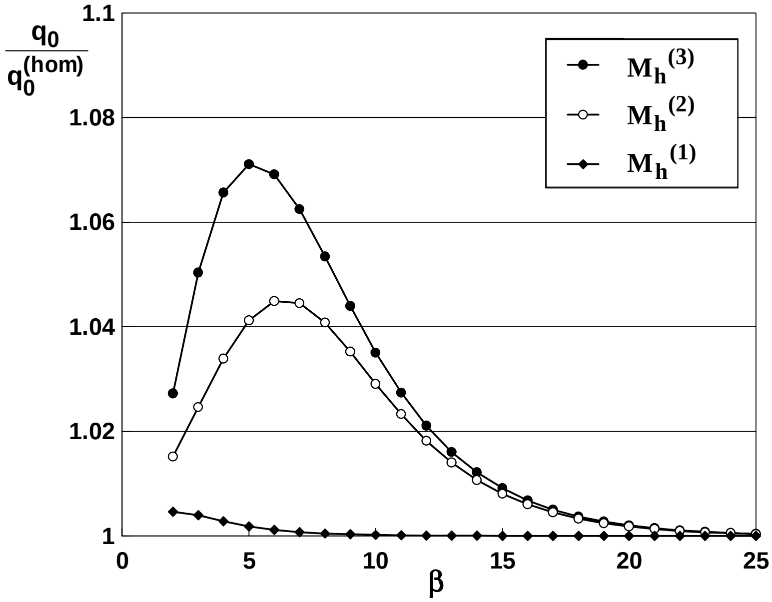

Fig. 4. Effect of β on the magnitude of allowable load, for the three load distributions in (5.1). $\langle S'^2 \rangle = 1/12$, $\lambda = 0.1$.

which has the highest stresses, has the major contribution to the failure probability. Then, notice that the highest stresses in all three cases are close to $x = 1$, where the local distribution in that region has a very high gradient for $M_h^{(1)}$ and very small one for $M_h^{(3)}$. High gradients near the maximum point means that the highest stress region is very concentrated, and therefore the beam strength is practically affected by the morphology of that small region only. This is equivalent to a beam with larger grains, which is more morphology sensitive.

Figs. 3 and 4, show that morphology effects diminish for $\beta \to 0$ and $\beta \to \infty$, as expected. However, certain β values may possess reliability advantages, which are not negligible. To see the source of this advantage, the two morphology parts $F_{\mathrm{b}}^{(2)}$ and $F_{\mathrm{b}}^{(3)}$ are plotted separately for the first two cases as seen in Fig. 5. We see that the two have opposite effects on reliability, and that $F_{\mathrm{b}}^{(3)}$ is more dominant here. From (3.23d), it is seen that $F_{\mathrm{b}}^{(3)}$, *which is the net SMC effect*, is proportional to $\theta_{,S}$ at $\langle S \rangle$. Thus, in the absence of such correlation ($\theta_{,S} = 0$), a completely opposite effect is predicted, as found also in a previous study (Altus and Givli, 2003).

5.2. Exponential two points correlation function

The effect of heterogeneity on strength in the examples above was calculated using (4.5) under the approximation of a small correlation length ($\lambda \ll 1$), in order to explore the "β effect". In the following we study the heterogeneity effect for the whole range of compliance correlation lengths, by choosing a common two-point probability function:

$$\langle S_1' S_2' \rangle = \langle S'^2 \rangle \cdot \exp\left(-\frac{|x_1 - x_2|}{\lambda/2} \right), \tag{5.2}$$

where λ in (5.2) fulfills the definition in (3.36). Fig. 6 shows a comparison between an accurate (by numerical integration) solution based on (5.2) and an analytical solution based on (4.5) for $\lambda \ll 1$, for a

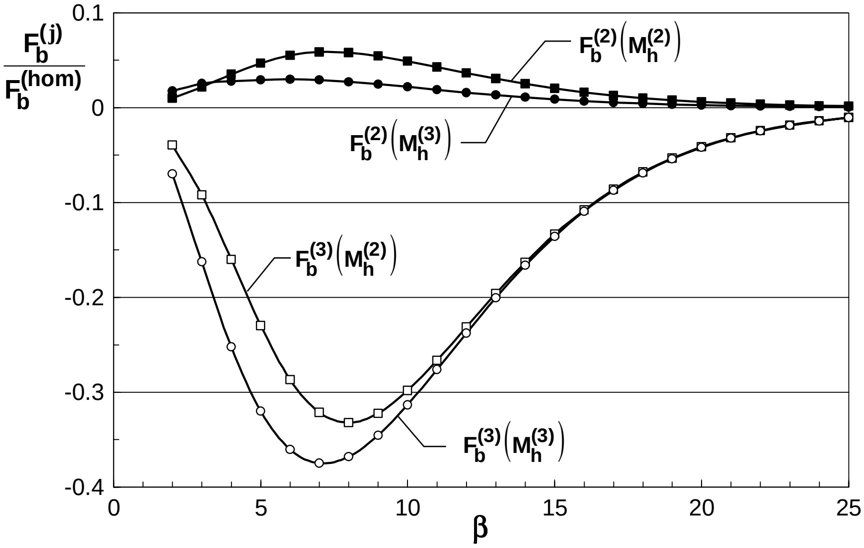

Fig. 5. Relative contribution of $F_b^{(2)}$ and $F_b^{(3)}$ to the failure probability of the beam for two types of loading distributions (5.1).

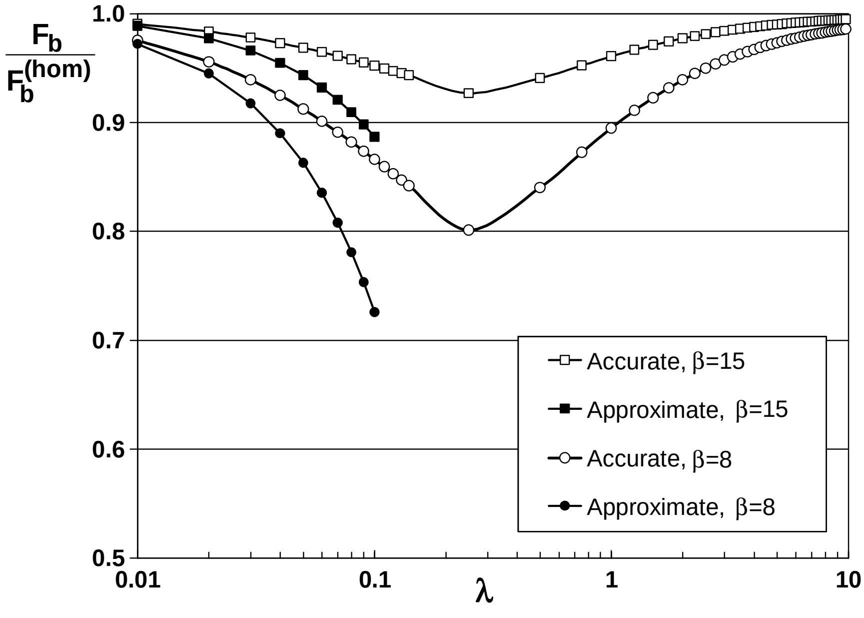

Fig. 6. Heterogeneity effects on failure probability, as a function of compliance correlation length for two values of β. Accurate ((3.23) and (5.2)), and approximate (4.6) solutions. $\langle S'^2 \rangle = 1/12$, uniform distribution of external load.

uniform load distribution. The logarithmic scale is chosen for clear exposition to the whole range of λ. As expected, an extremum value of λ exists, for which the reliability of the beam is the highest. It is also seen that the analytical expression is a good approximation for $\lambda \ll 1$, but its accuracy is also dominated by β. Other types of external loading may yield different values, but the overall behavior is expected to be similar.

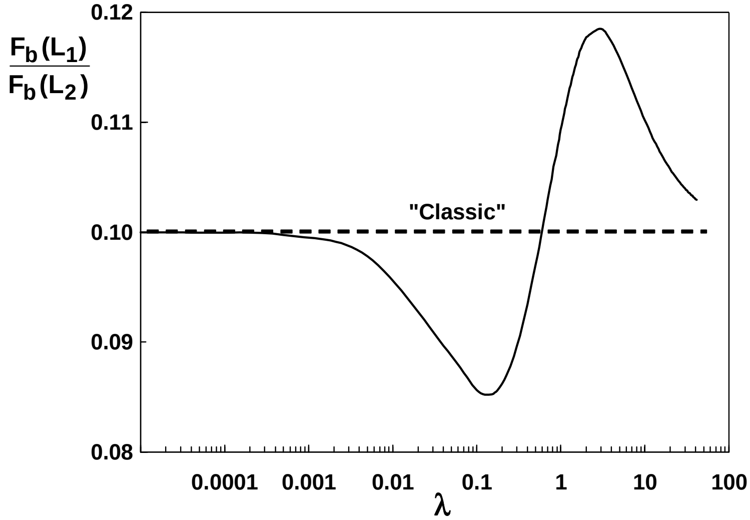

Fig. 7. Morphology dependent size effect (3.39) for $L_1/L_2 = 0.1$, $\beta = 8$ compared to the classical value.

Notice that when $\lambda \to \infty$ the internal bending moment is not random, and the problem reduces to a statically determinate case, with a solution given by (2.25), identical to the homogeneous ($\lambda \to 0$) case in (4.5).

Using (3.39), the morphology influence on the size effect of F_b is examined for $M_h^{(1)}$ (uniform loading distribution) as seen in Fig. 7. $L_1/L_2 = 1/10$ is chosen for demonstration purpose, so the line $F_b^{(L_2)}/F_b^{(L_1)} = 0.1$ represents the "classical" (power law) size effect (Hansen and Roux, 2000). It is seen that the morphology effect is not negligible and that the correction may be either positive or negative, depending on the correlation length λ.

6. Discussion and conclusions

The effect of SMC on strength reliability was studied analytically by the FPM. The reliability was found to be composed of three types of contributions, which add up to the final estimation: (I) calculation of a homogeneous beam of average properties, (II) correction corresponding to beam heterogeneity for non-correlated strength-moduli, and (III) specific SMC effect. For the examples studied, part III was more dominant than II.

Other major conclusions from this study are

1. The FPM proves to be a powerful tool for analyzing stochastically heterogeneous materials. Its accuracy is limited only by the number of point probabilities used and has no stability problems for small "element size" as stochastic finite elements do (see also Altus and Totry (2003) for a buckling problem).
2. Exploiting the full morphological information of one and two point probability data, in addition to the common $\langle S\rangle$ and $\langle SS\rangle$ values, increases the accuracy of the solution. However, the ability to do so depends on the specific problem.
3. A positive SMC, increases the overall reliability of the heterogeneous beam, when compared to a structure without such correlation, and having the same statistical properties. On the other hand, the heterogeneity

itself has a negative effect on strength reliability. Therefore, the overall effect depends on the specific loading shape.

4. An optimum grain-size was found (~0.3 L) for the example studied, for which the average failure probability is minimal, with a value, which strongly depends on β. This type of information may be useful for future design with heterogeneous materials.
5. A non-classical morphology dependent size effect for strength was found, which deviate considerably from the classical one.

Finally, the interaction between material morphology, loading geometry and strength distribution function is complex and non-linear even for a case of a simple beam, having one degree of indeterminacy. This can explain the difficulty in finding predictive tools for more general microstructures, and emphasizes the need to develop new design tools for stochastically heterogeneous materials.

Acknowledgement

The research was partially supported by the Israeli Science Foundation.

References

Altus, E., 2001. Statistical modeling of heterogeneous microbeams. Int. J. Solids Struct. 38, 5915–5934.
Altus, E., Givli, S., 2003. Strength reliability of statically indeterminate heterogeneous beams. Int. J. Solids Struct. 40, 2069–2083.
Altus, E., Totry, M.E., 2003. Buckling of stochastically heterogeneous beams, using a functional perturbation Method. IJSS. In print.
Barbe, F., Decker, L., Jeulin, D., Cailletaud, G., 2000a. Intergranular and intragranular behavior of polycrystalline aggregates. Part I: F.E. model. Int. J. Plast. 17, 513–536.
Barbe, F., Forest, S., Cailletaud, G., 2000b. Intergranular and intragranular behavior of polycrystalline aggregates. Part II: Results. Int. J. Plast. 17, 537–563.
Beran, M.J., 1967. Statistical Continuum Theories. McGraw-Hill. pp. 62–64.
Coquard, P., Boistelle, R., Amathieu, L., Barriac, P., 1994. Hardness, elasticity modulus and flexion strength of dry set plaster. J. Mater. Sci. 29, 4611–4617.
Davidge, R.W., 1979. Mechanical Behavior of Ceramics. Cambridge.
Elishakoff, I., 1983. Probabilistic Methods in the Theory of Structures. Wiley.
Frantziskonis, G., Breysse, D., 2003. Influence of soil variability in differential settlements of structures. Comput. Geotech. 30, 217–230.
Frantziskonis, G., Renaudin, P., Breysse, D., 1997. Heterogeneous solids—Part I: Analytical and numerical 1-D results on boundary effects. Eur. J. Mech. A/Solids 16 (3), 409–423.
Greek, S., Ericson, F., Johansson, S., Furtsch, M., Rump, A., 1999. Mechanical characterization of thick polysilicon films: Young's modulus and fracture strength evaluated with microstructures. J. Micromech. Microeng. 9, 151–245.
Hansen, A., Roux, S., 2000. Statistics toolbox for damage and fracture. In: Krajcinovic, D., Van Meir, J. (Eds.), CISM 410: Damage and Fracture of Disordered Materials. Springer, pp. 17–101.
Harder, J., 1999. A crystallographic model for the study of local deformation processes in polycrystals. Int. J. Plast. 15, 605–624.
Hashin, Z., Shtrikman, S., 1962. On some variational principles in anisotropic and nonhomogeneous elasticity (Part I). J. Mech. Phys. Solids 10, 335–342.
Herrmann, H.J., Roux, S. (Eds.), 1990. Statistical Model for the Fracture of Disordered Media. Elsevier.
Hill, R., 1952. The elastic behavior of a crystalline aggregate. Proc. Phys. Soc. (Lond) A65, 349.
Jeulin, D., 1993. Morphological models for fracture statistics. In: Proc. 7th Int. Symp. Continuum Models of Discrete Systems, 1992, vols. 123–125. Materials Science Forum, pp. 505–514.
Jones, P.T., Johnson, G.C., Howe, R.T., 1999. Statistical characterization of fracture of brittle MEMS materials. In: Proc. MEMS Reliability for Critical and Space Applications, vols. 20–29. Soc. Of Photo-Optical Instr. Eng., Bellingham, WA, USA.
Kanwal, R.P., 1983. Generalized Functions, Theory and Technique. Academic Press.
Keller, T.S., Hansson, T.H., Abram, A.C., Spengler, D.M., Panjabi, M.M., 1989. Regional variations in the compressive properties of lumbar vertebral trabeculae. SPINE 14 (9), 1012–1019.

Keller, T.S., Mao, Z., Spengler, D.M., 1990. Young's modulus, bending strength, and tissue physical properties of human compact bone. J. Orthop. Res. 8, 592–603.

Keyak, J.H., Lee, I.Y., Skinner, H.B., 1994. Correlations between orthogonal mechanical properties and density of trabecular bone: use of different densitometric measures. J. Biomed. Mater. Res. 28, 1329–1336.

Kim, H.S., Guifang, T., Kim, J.Y., 2002. Clayware mechanical properties porosity dependent. Am. Ceram. Soc. Bull. 81 (5), 20–25.

Kingery, W.D., Bowen, H.K., Uhlmann, D.R., 1976. Introduction to Ceramics. Wiley.

Kröner, E., 1986. Statistical modelling. In: Gittus, J., Zarka, J. (Eds.), Modeling Small Deformation of Polycrystals. Elsevier, pp. 229–291 (Chapter 8).

Lin, Y.K., Cai, G.Q., 1995. Probabilistic Structural Dynamics. McGraw-Hill.

Mason, T.A., Adams, B.L., 1999. Use of microstructural statistics in predicting polycrystalline material properties. Metall. Mater. Trans. A 30, 969–979.

Mirfendereski, D., Kiureghian, A., Ferrari, M., 1992. Analysis of microfabricated textured multicrystalline beams: II. Probabilistic approach. Mat. Res. Symp. Proc. 276, 97–101.

Namazu, T., Isono, Y., Tanaka, T., 2000. Evaluation of size effect on mechanical properties of single crystal silicon by nanoscale bending test using AFM. J. Micromech. Syst. 9 (4), 450–459.

Nanjangus, S.C., Brezny, R., Green, D.J., 1995. Strength and Young's modulus behavior of a partially sintered porous alumina. J. Am. Ceram. Soc. 78 (1), 266–298.

Reuss, A., 1929. Berchnung der Fliessgrenze von Mischkristallen auf Grund der Plastizitatsbedingung fur Einkristalle. Z. Ang. Math. Mech. 9, 49.

Sharpe, W.N., Yuan, B., Vaidynathan, R., Edwards, R., 1999. Measurements of Young modulus, Poisson ratio and Tensile strength of polysilicon. IEEE, 424–429.

Snead, L.L., Osborne, M., More, K.L., 1995. Effects of radiation on SiC-based nicalon fibers. J. Mater. Res. 10 (3), 736–747.

Starzewski, M.O., Stahl, D.C., 2000. Random fiber networks and special elastic orthotropy of paper. J. Elasticity 60, 131–149.

Voigt, W., 1887. Theoretische Studien uber die Elastizitatsverha-Itnisse der Krystalle. Abh. Kgl. Ges. Wiss. Gottingen, Math. Kl 34 (1), 47.